Deterministic Smooth Number Cover and Lemoine Obstructions

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We derive a deterministic covering theorem: for any even $N \geq 8$, the interval $[1, N]$ is covered by Y-smooth numbers and their complements with respect to $N+1$ provided $Y \geq \lceil N/3 \rceil$. We further identify the unique obstruction type just below this threshold: in the window $\lfloor N/5 \rfloor < Y < \lfloor N/3 \rfloor$, we prove the cover fails if and only if $N+1$ admits a Lemoine representation $N+1 = p+2q$ with primes $p, q > Y$. Motivated by computation, we propose a strengthened (well-balanced) Lemoine-type conjecture asserting that such representations can be chosen with both primes located within a polylogarithmic distance of $N/3$.

Keywords: Smooth numbers; additive covering; difference bases; Lemoine's conjecture; linear Diophantine obstructions.

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1. Introduction

The distribution of smooth numbers—integers free of large prime factors—is a cornerstone of analytic number theory, underpinning the analysis of factorization algorithms and the structure of friable sets. Classical theory, dating back to Dickman [1] and de Bruijn [2], through to more contemporary scholarship [3,4], provides precise asymptotic estimates for $\Psi(x, y)$, the count of y-smooth integers up to x. These results, however, are probabilistic: they describe densities and likelihoods, but rarely offer deterministic guarantees about specific intervals.

In this paper, we depart from the asymptotic view to investigate a strict, deterministic covering structure. We ask a simple combinatorial question: For a given horizon N, how large must the smoothness bound Y be to guarantee that for every integer $k \in [1, N]$, either k or its complement $N+1-k$ is Y-smooth? We term this the **Covering Property**, denoted $C(N, Y)$.

Main results. Our first result establishes an explicit, unconditional threshold: the covering property holds whenever $Y \geq \lceil N/3 \rceil$. This result is derived from a "Failure Constraint Lemma," which maps any failure of the cover to a solution of a constrained linear Diophantine equation. The Diophantine framework reveals a surprising connection to classical additive number theory in the "near-critical" regime. We show that just below the deterministic threshold—specifically in the

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window $\lfloor N/5 \rfloor < Y < \lfloor N/3 \rfloor$—the cover fails *if and only if* $N+1$ admits a specific Lemoine representation, $N+1 = p+2q$, with primes $p, q > Y$.

Motivated by computational evidence up to 10^9, we formulate a "well-balanced Lemoine Conjecture". For future consideration, we outline a partition-based hierarchy that organizes higher-order failure modes as Y decreases further.

2. Setup and Definitions

Let $N \geq 8$ be an even integer and $Y \in \mathbb{N}$ be a smoothness bound. Let $P^+(n)$ denote the largest prime factor of a positive integer n. We say that an integer n is *Y-smooth* if $P^+(n) \leq Y$.

We define the **Covering Property** $C(N, Y)$ as follows. For every integer $k \in [1, N]$, at least one of the following holds:

(1) k is Y-smooth, i.e. $P^+(k) \leq Y$;
(2) the complement $N+1-k$ is Y-smooth, i.e. $P^+(N+1-k) \leq Y$.

Equivalently, the property $C(N, Y)$ fails if and only if there exists a *failure integer* $k \in [1, N]$ such that

$$P^+(k) > Y \qquad \text{and} \qquad P^+(N+1-k) > Y.$$

3. The Failure Constraint Lemma

We first establish a general lemma that characterizes all possible obstructions to the cover. This lemma provides the Diophantine constraints satisfied by any failure.

Lemma 3.1 (Failure Constraint Lemma). *Let $N \geq 8$ be even and $Y \in \mathbb{N}$ with $Y \geq 2$. Then the covering property $C(N, Y)$ fails if and only if there exist primes $p, q > Y$ and positive integers a, b such that*

$$ap + bq = N + 1 \tag{3.1}$$

with

(1) a and b have opposite parity (one even, one odd);
(2) the coefficient sum is bounded by

$$3 \leq a + b \leq \left\lfloor \frac{N+1}{Y+1} \right\rfloor. \tag{3.2}$$

Proof. *(Failure implies Diophantine structure.)* Assume $C(N, Y)$ fails. Then there exists $k \in [1, N]$ such that

$$P^+(k) > Y \qquad \text{and} \qquad P^+(N+1-k) > Y.$$

Let $p = P^+(k)$ and $q = P^+(N+1-k)$. Since $Y \in \mathbb{N}$ and $P^+(k), P^+(N+1-k) > Y$, we have

$$p, q \geq Y + 1.$$

Because p and q are prime factors of k and $N+1-k$, respectively, there exist integers $a, b \geq 1$ such that

$$k = ap, \qquad N+1-k = bq.$$

Summing these gives the linear Diophantine equation

$$ap + bq = N + 1.$$

For the parity constraint, note that N is even, so $N+1$ is odd. Since $Y \geq 2$, we have $Y+1 \geq 3$, so $p, q \geq 3$ and hence p and q are odd primes. For $ap+bq$ to be odd, the terms ap and bq must have opposite parity. Because p and q are odd, this forces the coefficients a and b to have opposite parity. In particular $a+b$ is odd, and since $a, b \geq 1$ we have

$$a + b \geq 1 + 2 = 3.$$

For the upper bound on $a+b$, we use $p, q \geq Y+1$ to obtain

$$N + 1 = ap + bq \geq a(Y+1) + b(Y+1) = (a+b)(Y+1).$$

Dividing by $Y+1$ yields

$$a + b \leq \frac{N+1}{Y+1},$$

and since $a+b$ is an integer we obtain

$$a + b \leq \left\lfloor \frac{N+1}{Y+1} \right\rfloor.$$

This establishes (3.1)–(3.2).

Conversely, suppose there exist primes $p, q > Y$ and integers $a, b \geq 1$ satisfying (3.1). Define

$$k := ap.$$

Then $1 \leq k \leq N$ because $k > 0$ and

$$N + 1 - k = N + 1 - ap = bq > 0,$$

so $k < N+1$. Moreover,

$$P^+(k) \geq p > Y, \qquad P^+(N+1-k) = P^+(bq) \geq q > Y,$$

so k is a failure integer for $C(N, Y)$. □

4. The Deterministic Bound (Sufficiency)

Lemma 3.1 yields a simple sufficient condition under which no failure can occur.

Theorem 4.1 (Deterministic Covering Threshold). *Let $N \geq 8$ be an even integer. If*

$$Y \geq \left\lceil \frac{N}{3} \right\rceil,$$

then the covering property $C(N, Y)$ holds.

Proof. Assume, for contradiction, that $C(N, Y)$ fails for some even $N \geq 8$ and some

$$Y \geq \left\lceil \frac{N}{3} \right\rceil.$$

Since $N \geq 8$, we have $\lceil N/3 \rceil \geq 3$, so $Y \geq 3$ and Lemma 3.1 applies. In particular, there exist integers $a, b \geq 1$ and primes $p, q > Y$ satisfying (3.1)–(3.2).

From $Y \geq N/3$ we obtain

$$Y + 1 \; \geq \; \frac{N}{3} + 1.$$

Consider the quantity $\dfrac{N+1}{Y+1}$. Using $Y + 1 \geq (N/3) + 1$ we obtain

$$\frac{N+1}{Y+1} \; \leq \; \frac{N+1}{(N/3)+1} = \frac{3(N+1)}{N+3} < 3$$

for all $N > 0$. Hence

$$a + b \leq \left\lfloor \frac{N+1}{Y+1} \right\rfloor \leq 2.$$

This contradicts the lower bound $a + b \geq 3$ from Lemma 3.1. Therefore no such failure can occur, and $C(N, Y)$ holds.

In the appendix, we extend this work straightforwardly to show a difference basis corollary, and then a global difference representation: a proof that every integer may be represented as the difference of two sufficiently smooth numbers. Specifically, we show that for any $k \in \mathbb{Z}$, there exist S_1, S_2 with $P^+(S_i) \leq |k|/3 + 2$ such that $k = S_1 - S_2$. □

5. The Near-Critical Regime

We now analyze the behavior just below the deterministic threshold. In a specific window where the coefficient sum $a + b$ is constrained to be less than 5, the failure mode is uniquely determined.

Proposition 5.1 (Near-Critical Lemoine Regime). *Let $N \geq 8$ be even and let $Y \in \mathbb{N}$ satisfy*

$$\left\lfloor \frac{N}{5} \right\rfloor < Y < \left\lfloor \frac{N}{3} \right\rfloor.$$

Then the covering property $C(N, Y)$ fails if and only if there exist primes $p, q > Y$ such that

$$p + 2q = N + 1.$$

Proof. Assume first that $C(N, Y)$ fails. Since $N \geq 8$, we have $\lfloor N/5 \rfloor \geq 1$. Thus $Y > 1$ implies $Y \geq 2$, so Lemma 3.1 applies. Any failure then yields primes $p, q > Y$ and integers $a, b \geq 1$ with

$$ap + bq = N + 1, \qquad 3 \leq a + b \leq \left\lfloor \frac{N+1}{Y+1} \right\rfloor,$$

and $a + b$ odd.

From the lower bound on Y we have $Y > N/5$. Thus

$$\frac{N+1}{Y+1} < \frac{N+1}{(N/5)+1} = \frac{5(N+1)}{N+5} < 5$$

for all $N > 0$. Therefore

$$a + b \leq \left\lfloor \frac{N+1}{Y+1} \right\rfloor \leq 4.$$

Combining this with $a + b \geq 3$ and $a + b$ odd forces

$$a + b = 3.$$

The only positive integer solutions to $a+b = 3$ are $\{a, b\} = \{1, 2\}$. Hence any failure must satisfy

$$ap + bq = N + 1$$

with $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$. Equivalently,

$$p + 2q = N + 1 \quad \text{or} \quad 2p + q = N + 1.$$

Renaming the roles of p and q if necessary, we obtain a representation of the form

$$p + 2q = N + 1$$

with primes $p, q > Y$.

Conversely, suppose there exist primes $p, q > Y$ such that

$$p + 2q = N + 1.$$

Set $a = 1$, $b = 2$ and define $k = 2q$. Then

$$N + 1 - k = N + 1 - 2q = p.$$

We have $1 \leq k \leq N$ because $p > 0$ and $p + 2q = N + 1$ implies $2q \leq N$. Moreover

$$P^+(k) = q > Y, \qquad P^+(N + 1 - k) = P^+(p) = p > Y.$$

Thus k is a failure integer for $C(N, Y)$, and the cover fails. □

Remark 5.2. Proposition 5.1 links the covering problem in this parameter window to the classical Lemoine equation

$$p + 2q = M,$$

with the additional constraint that the primes p, q lie above the smoothness threshold Y. In the near-critical regime $\lfloor N/5 \rfloor < Y < \lfloor N/3 \rfloor$, the cover fails exactly when $N + 1$ admits such a Lemoine partition with components larger than Y.

5.1. *A well-balanced Strengthening of Lemoine (Conjectural)*

Proposition 5.1 identifies the *only* obstruction to the covering property in the near-critical window $\lfloor N/5 \rfloor < Y < \lfloor N/3 \rfloor$: namely, the existence of primes $p, q > Y$ satisfying $N + 1 = p + 2q$. This invites a sharper question than Lemoine's original conjecture: how close to the boundary $M/3$ can one force such a representation to lie?

Definition 5.3 (Critical Lemoine threshold). Let M be an odd integer. Define the *critical Lemoine threshold*

$$Y_{\text{crit}}(M) \ := \ \max\Big\{ \min(p, q) \ : \ M = p + 2q, \quad p, q \text{ prime} \Big\},$$

with the convention that $Y_{\text{crit}}(M) = 0$ if no such representation exists. Note that necessarily $Y_{\text{crit}}(M) \leq \lfloor M/3 \rfloor$.

Remark 5.4 (Two symmetric branches). If $q \leq \lfloor M/3 \rfloor$, then $p = M - 2q \geq \lfloor M/3 \rfloor$ and $\min(p, q) = q$. Thus the best witness in this *lower* branch is

$$Y_{\text{lower}}(M) \ := \ \max\Big\{ q \leq \Big\lfloor \frac{M}{3} \Big\rfloor \ : \ q \text{ prime and } \ M - 2q \text{ prime} \Big\}.$$

If $q > \lfloor M/3 \rfloor$, then $p = M - 2q < \lfloor M/3 \rfloor$ and $\min(p, q) = p$, so the best witness in the *upper* branch can be written as

$$Y_{\text{upper}}(M) \ := \ \max\Big\{ p \leq \Big\lfloor \frac{M}{3} \Big\rfloor \ : \ p \text{ prime and } \ \frac{M - p}{2} \text{ prime} \Big\}.$$

In particular, $Y_{\text{crit}}(M) = \max\{Y_{\text{lower}}(M), Y_{\text{upper}}(M)\}$ whenever at least one branch is nonempty. Moreover, if $\min(p, q) \geq \lfloor M/3 \rfloor - d$ for some witness $M = p + 2q$, then both primes p and q lie within $O(d)$ of $M/3$ (with explicit constants depending only on which branch the witness occupies).

Conjecture 5.5 (well-balanced Lemoine (polylogarithmic form)). *There exist constants $C > 0$ and $\kappa > 0$ such that for all sufficiently large odd integers M,*

$$\Big\lfloor \frac{M}{3} \Big\rfloor - Y_{\text{crit}}(M) \ \leq \ C \, (\log M)^{\kappa}.$$

Equivalently, for all sufficiently large odd M there exist primes p, q such that $M = p + 2q$ and

$$\min(p, q) \ \geq \ \Big\lfloor \frac{M}{3} \Big\rfloor - C(\log M)^{\kappa}.$$

Remark 5.6 (Empirical evidence up to 10^9). The Lemoine conjecture itself has been verified up to 10^{13} [7]. More broadly, Juhász et al. also test two–prime partitions of the form $n = m_1p + m_2q$ across many small coefficient pairs, providing computational context for the partition hierarchy considered here. An exhaustive scan over all even $10^2 \leq N \leq 10^9$ (equivalently all odd $M = 10^2 + 1 \leq N+1 \leq 10^9 + 1$) found at least one representation $M = p + 2q$ in every case, supporting the view that a subset of Lemoine witnesses typically occur in a narrow strip near the $N/3$ boundary. Defining

$$t(N) \;:=\; \left\lfloor \frac{N}{3} \right\rfloor - Y_{\text{crit}}(N+1),$$

the largest value observed over the full scan was $t(N) = 3{,}057$, first attained at $N = 525{,}277{,}308$ (where $M = N+1 = 525{,}277{,}309 = 175{,}098{,}551 + 2 \cdot 175{,}089{,}379$). Across the full range, both window averages of $t(N)$ and the record envelope appear to follow polylogarithmic growth in N. Empirically, the window-averaged behavior is consistent with an exponent between 2.2 and 2.3, while the record envelope grows faster; however, the record sequence is comparatively sparse, and we do not treat a fitted exponent for the envelope as stable at this scale. Figure 1 provides a visual summary.

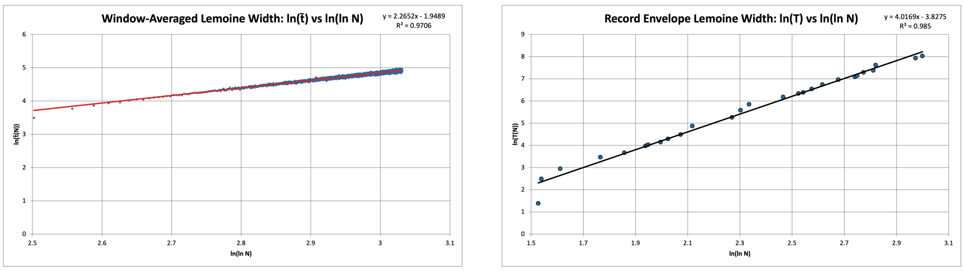

Fig. 1. Empirical thin-strip widths for Lemoine witnesses up to 10^9. **Left:** window-averaged values of $t(N) = \lfloor N/3 \rfloor - Y_{\text{crit}}(N+1)$ computed over consecutive blocks of even N (with block size 100,000 in the attached data), exhibiting polylogarithmic growth with an estimated exponent near 2.2–2.3 over this range. **Right:** record-breaking (envelope) values of $t(N)$, which grow faster than the window averages; the envelope data are comparatively sparse, so any fitted exponent at this scale should be treated as indicative rather than definitive.

6. Outlook: The Partition Hierarchy

Partition Hierarchy. As Y decreases further below the deterministic threshold, the upper bound $\left\lfloor \frac{N+1}{Y+1} \right\rfloor$ from Lemma 3.1 increases, allowing larger odd values for the coefficient sum

$$K := a + b.$$

This suggests a hierarchical structure for potential failures indexed by odd integers $K \geq 3$.

The first two tiers correspond to the sharp $N/3$ threshold proved above and the near-critical Lemoine window of Proposition 5.1; higher tiers should be regarded as heuristic structure rather than established theorems.

Conjecture 6.1 (Cumulative Partition Hierarchy). *Let N be even and Y a smoothness bound. Heuristically, when Y drops below the order of N/K for an odd integer $K \geq 3$, the set of potential failure modes* ***expands cumulatively****. The admissible Diophantine obstructions include all linear combinations*

$$ap + bq = N + 1$$

with primes $p, q > Y$ where the coefficient sum $a + b$ partitions K, in addition to all previously active partitions.

We summarize the resulting cumulative picture in Table 1.

Table 1. Cumulative hierarchy of failure tiers. As the divisor increases, new failure modes (bold) are added to the existing set.

Tier	**Smoothness** ($Y \approx$)	**Max Sum** (K)	**Active Failure Partitions** $\{a, b\}$
1	$N/3$	3	$\{\mathbf{1}, \mathbf{2}\}$
2	$N/5$	5	$\{1, 2\} \cup \{\mathbf{1}, \mathbf{4}\}, \{\mathbf{2}, \mathbf{3}\}$
3	$N/7$	7	$\{1, 2\} \cdots \cup \{\mathbf{1}, \mathbf{6}\}, \{\mathbf{2}, \mathbf{5}\}, \{\mathbf{3}, \mathbf{4}\}$
4	$N/9$	9	$\cdots \cup \{\mathbf{1}, \mathbf{8}\}, \{\mathbf{2}, \mathbf{7}\}, \{\mathbf{3}, \mathbf{6}\}, \{\mathbf{4}, \mathbf{5}\}$

Remark 6.2 (Damped Accumulation). While Conjecture 6.1 predicts a combinatorial expansion of failure modes, empirical evidence suggests that the aggregate failure rate grows only log-linearly with the divisor $D = N/Y$. This can be attributed to the decreasing asymptotic density of solutions for Diophantine equations with larger coefficients.

Study of the partitions appears to be a fertile direction for further work, along with sharpening Conjectures 5.5 and 6.1.

Appendix A. A Difference Basis Corollary

If the pivot $N+1$ is itself smooth, the covering property implies a simple difference-basis representation.

Corollary Appendix A.1. *Let $N \geq 8$ be even and let $Y \geq \lceil N/3 \rceil$. Suppose that $P^+(N+1) \leq Y$. Then every integer $k \in [1, N]$ can be represented as the difference* *of two Y-smooth numbers,*

$$k = S_1 - S_2.$$

Proof. By Theorem 4.1, the covering property $C(N, Y)$ holds for this N and Y. Thus for each $k \in [1, N]$, either k is Y-smooth or $N + 1 - k$ is Y-smooth.

Case A: k is Y-smooth. Write $k = 2k - k$. Since $Y \geq \lceil N/3 \rceil$ and $N \geq 8$, we have $\lceil N/3 \rceil \geq 3$, hence $Y \geq 3$. Thus if $P^+(k) \leq Y$, then

$$P^+(2k) = \max\bigl(P^+(k), 2\bigr) \leq Y,$$

so $2k$ is also Y-smooth. Thus we may take

$$S_1 = 2k, \qquad S_2 = k.$$

Case B: $N+1-k$ is Y-smooth. Let $S' = N+1-k$. By hypothesis $P^+(N+1) \leq Y$, so $N + 1$ is Y-smooth. Then

$$k = (N + 1) - S'$$

expresses k as the difference of the two Y-smooth numbers $S_1 = N+1$ and $S_2 = S'$. □

Appendix B. A Global Difference Representation via Expanding Horizons

The preceding corollary shows that, for a fixed even horizon N, the interval $[1, N]$ is a difference basis for the Y-smooth numbers (for suitable Y) provided the pivot $N + 1$ is itself Y-smooth. In this section we observe that for a large and explicit family of even horizons the pivot condition holds *deterministically*, and that this can be upgraded to a global statement: every positive integer can be represented as the difference of two sufficiently smooth numbers, with a smoothness parameter bounded by $|k|/3 + 2$.

We begin by isolating the special role of horizons whose pivot is divisible by 3.

Proposition Appendix B.1 (Pivot-smooth horizons). *Let $N \geq 8$ be even, and set*

$$Y := \left\lceil \frac{N}{3} \right\rceil.$$

Suppose that $N + 1$ is divisible by 3. Then:

(1) The covering property $C(N, Y)$ holds.
(2) The pivot $N + 1$ is Y-smooth.
(3) Consequently, every integer $k \in [1, N]$ can be written as

$$k = S_1 - S_2$$

with S_1, S_2 both Y-smooth.

Proof. The deterministic covering threshold (Theorem 4.1) implies that $C(N,Y)$ holds for every even $N \geq 8$ as soon as $Y \geq \lceil N/3 \rceil$. This gives (1).

For (2), write $N+1 = 3m$. Since $N \geq 8$, we have $N+1 \geq 9$, so $m \geq 3$. Every prime factor of $N+1$ is either 3 or a prime factor of m, hence

$$P^{+}(N+1) \;\leq\; \max\{3,\, P^{+}(m)\} \;\leq\; m \;=\; \frac{N+1}{3}.$$

On the other hand, if $N+1$ is divisible by 3, then $N \equiv 2 \pmod 6$, so in particular $3 \nmid N$ and

$$\frac{N+1}{3} = \left\lceil \frac{N}{3} \right\rceil = Y.$$

Thus $P^{+}(N+1) \leq Y$, i.e. $N+1$ is Y-smooth.

For (3), we now have an even $N \geq 8$, a smoothness bound $Y \geq \lceil N/3 \rceil$, the covering property $C(N,Y)$, and the pivot smoothness hypothesis $P^{+}(N+1) \leq Y$. Thus all the hypotheses of Corollary Appendix A.1 are satisfied, and we conclude that every $k \in [1,N]$ can be written as the difference of two Y-smooth integers. □

The divisibility condition on the pivot can be enforced by a simple congruence restriction on the horizon:

$$N \equiv 2 \pmod 6 \quad\Longleftrightarrow\quad \begin{cases} N \text{ is even,} \\ N+1 \equiv 0 \pmod 3. \end{cases}$$

Thus every even integer in the progression

$$N \in \{8, 14, 20, 26, 32, \dots\}, \qquad N \equiv 2 \pmod 6,$$

is a "pivot-smooth horizon" in the sense of the proposition.

We can now globalize the difference-basis phenomenon by allowing the horizon to depend on the integer we wish to represent.

Theorem Appendix B.2 (Global difference representation). *For every integer $k \in \mathbb{Z}$, there exist integers S_1, S_2 and a smoothness bound Y such that:*

(1) $k = S_1 - S_2$;
(2) both S_1 and S_2 are Y-smooth;
(3) for $k \neq 0$, Y can be chosen with the scale bound

$$Y \leq \frac{|k|}{3} + 2.$$

Proof. If $k = 0$, we may choose any $Y \geq 2$ and set $S_1 = S_2 = 2$. If $k < 0$, let $k' = -k > 0$. If $k' = S_1 - S_2$, then $k = S_2 - S_1$, preserving smoothness. Thus it suffices to consider $k \geq 1$.

The finitely many cases $1 \leq k \leq 7$ can be verified directly, so we assume $k \geq 8$.

Let

$$L := \max\{8, k\}$$

and consider the six consecutive integers

$$L,\ L+1,\ L+2,\ L+3,\ L+4,\ L+5.$$

Exactly one of these is congruent to $2 \pmod 6$; call that choice N. Then

$$N \geq L \geq k, \qquad N \equiv 2 \pmod 6,$$

and we trivially have $N \leq L+5$, so in particular $k \leq N \leq k+5$ whenever $k \geq 8$.

Define

$$Y := \left\lceil \frac{N}{3} \right\rceil.$$

By construction $N \geq 8$ is even and $N+1$ is divisible by 3, so the horizon N satisfies the hypotheses of the pivot-smooth proposition above. Therefore:

- the covering property $C(N, Y)$ holds;
- the pivot $N+1$ is Y-smooth;
- every $k' \in [1, N]$ can be represented as the difference of two Y-smooth numbers.

Since our original integer k lies in $[1, N]$, we obtain integers S_1, S_2 with $k = S_1 - S_2$ and $P^+(S_1), P^+(S_2) \leq Y$, as claimed.

To establish the bound on Y, note that $N \equiv 2 \pmod 6$ implies $N = 6m+2$ for some integer m. Then

$$Y = \left\lceil \frac{6m+2}{3} \right\rceil = \left\lceil 2m + \frac{2}{3} \right\rceil = 2m+1.$$

From $N \leq k+5$, we have $6m+2 \leq k+5$, or $k \geq 6m-3$. Thus

$$\frac{k}{3} \geq \frac{6m-3}{3} = 2m-1.$$

Comparison with Y yields

$$Y - \frac{k}{3} \leq (2m+1) - (2m-1) = 2.$$

Hence $Y \leq \frac{k}{3} + 2$. □

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